LOGIC BLOOM · JEE MATHEMATICS

Quadratic Equations
Graphs · Roots · Range

Chapter · **Quadratic Equations** · Class 11 Mathematics

Graph of quadratics, location of roots, range & applications

Theory | Diagrams | Worked Applications | Main + Advanced

THE THREE TOOLS

$D = b^2 - 4ac$ · $a \cdot f(k)$ sign · vertex $(-b/2a, -D/4a)$

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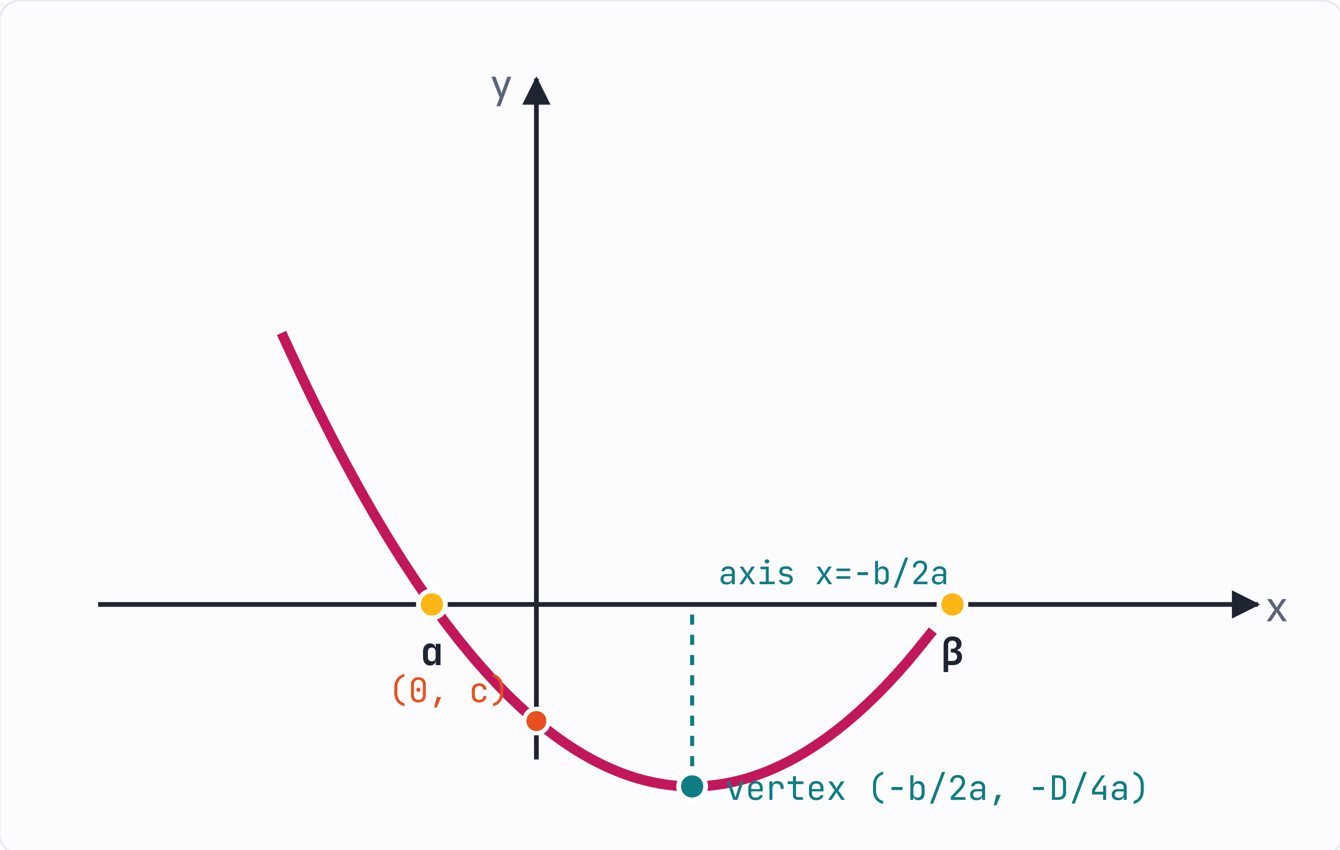
Contents

1 The Quadratic Function and Its Graph

A quadratic function is $f(x) = ax^2 + bx + c$ with $a \neq 0$. Its graph is a parabola. Almost every JEE question on this chapter is really a question about the *shape* and *position* of this parabola, so reading the graph fluently is the whole game.

```
f(x) = ax² + bx + c  (a ≠ 0)  ·  D = b² - 4ac
roots: x = [-b ± √D] / 2a  ·  vertex: (-b/2a, -D/4a)
sum α+β = -b/a  ·  product αβ = c/a
```



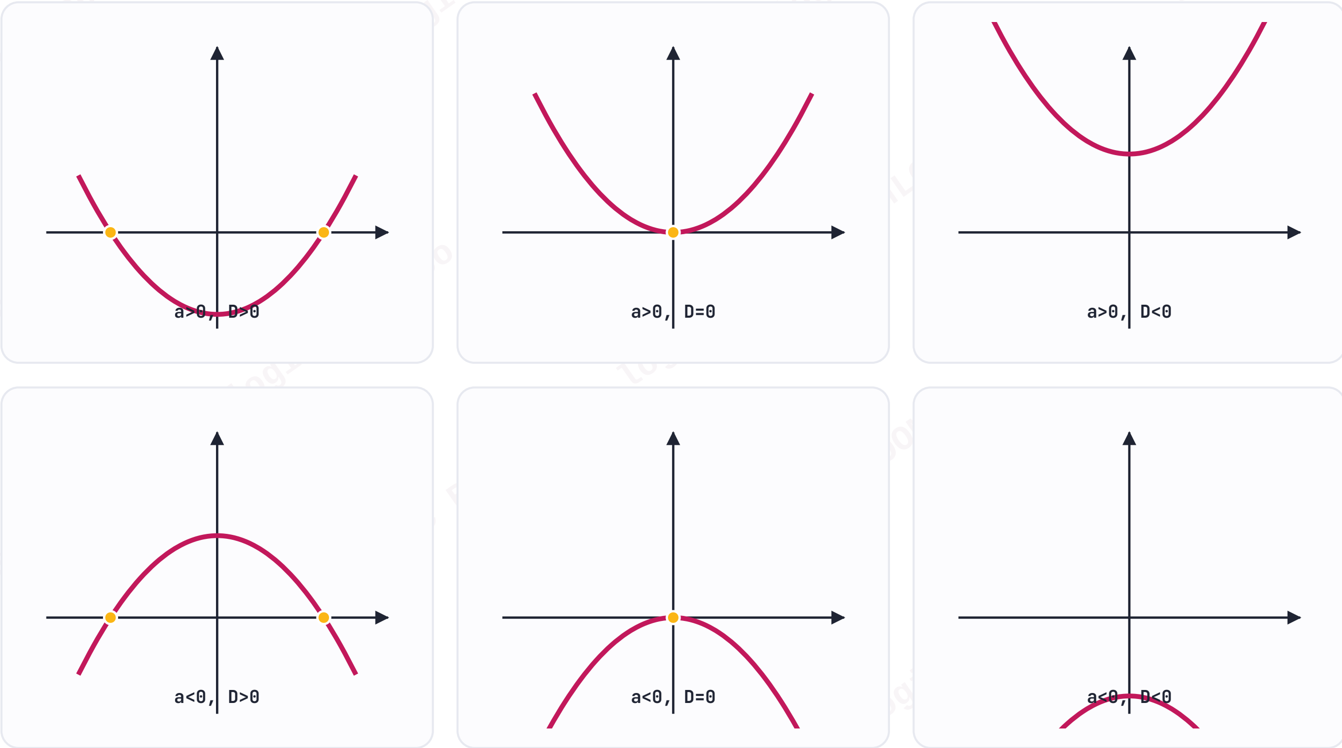
Anatomy of a parabola: the roots α, β (x-intercepts), the vertex (−b/2a, −D/4a), the axis of symmetry x = −b/2a, and the y-intercept (0, c).

READING A AND D OFF THE GRAPH

Two signs control everything. The sign of **a** fixes the opening: $a > 0$ opens upward (a "valley", vertex is a minimum), $a < 0$ opens downward (a "hill", vertex is a maximum).

The sign of the **discriminant D** fixes how the parabola meets the x-axis: $D > 0$ cuts it at two points (two distinct real roots), $D = 0$ touches it (one repeated root, vertex on the axis), $D < 0$ misses it entirely (no real root).

THE SIX FUNDAMENTAL CASES



The complete sign chart. Top row $a > 0$ (opens up); bottom row $a < 0$ (opens down). Left to right: $D > 0$ (two roots), $D = 0$ (one root), $D < 0$ (no real root).

SIGN OF F(X) — THE RESULT EVERYTHING ELSE RESTS ON

If $D < 0$, then $f(x)$ has the same sign as **a** for *every* real x (the parabola never crosses the axis). This single fact powers “always positive / always negative” problems:

- $ax^2+bx+c > 0$ for all $x \Leftrightarrow a > 0$ and $D < 0$.
- $ax^2+bx+c < 0$ for all $x \Leftrightarrow a < 0$ and $D < 0$.
- $ax^2+bx+c \geq 0$ for all $x \Leftrightarrow a > 0$ and $D \leq 0$.

If $D > 0$ with roots $\alpha < \beta$, then $f(x)$ has sign opposite to a *between* the roots and the same sign as a *outside* them — the classic “aαβ … αβa” sign pattern.

WORKED · ALWAYS POSITIVE

Find all m for which $(m-1)x^2 + 2(m-1)x + (m+2) > 0$ for all real x.

Two requirements. First the leading coefficient: $m - 1 > 0 \Rightarrow m > 1$. Second $D < 0$:

$$D = [2(m-1)]^2 - 4(m-1)(m+2) = 4(m-1)[(m-1) - (m+2)] = 4(m-1)(-3) = -12(m-1)$$

$D < 0 \Rightarrow -12(m-1) < 0 \Rightarrow m > 1$. Both conditions give $\mathbf{m > 1}$. (At $m = 1$ the expression collapses to the constant $3 > 0$, but it is no longer quadratic, so the strict quadratic answer is $m > 1$.)

2 Location of Roots

Here the roots are not asked for explicitly; instead we constrain *where* they sit relative to one or more reference numbers k, p, q. The entire toolkit is three quantities read off the parabola:

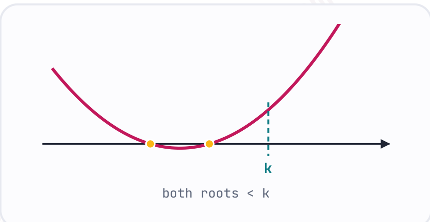
```
(i) Discriminant D ≥ 0 — guarantees the roots are real.
(ii) Sign of a·f(k) — tells you whether k is inside or outside the roots.
(iii) Position of the vertex -b/2a relative to k, p, q — pins the side.
```

WHY A·F(K) IS THE KEY QUANTITY

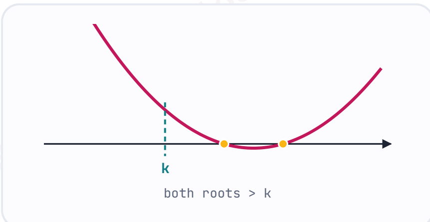
Multiplying by a removes the up/down ambiguity. Geometrically, **$a \cdot f(k) < 0$** means the point $x = k$ lies *strictly between* the two roots; **$a \cdot f(k) > 0$** means k lies outside the interval of the roots (either left of both or right of both). This one product replaces a page of casework.

THE FIVE STANDARD CONFIGURATIONS (WRITE A > 0; FOR A < 0 MULTIPLY F BY A)

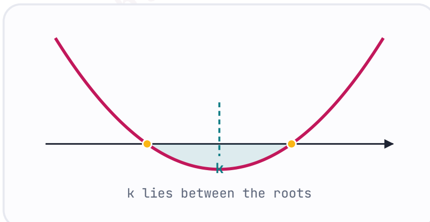
Required position of roots α, β	Conditions (all must hold)
Both roots **less than** k	$D \ge 0$ · $a \cdot f(k) > 0$ · $-b/2a < k$
Both roots **greater than** k	$D \ge 0$ · $a \cdot f(k) > 0$ · $-b/2a > k$
k **lies between** the roots ($\alpha < k < \beta$)	$a \cdot f(k) < 0$ (this alone forces $D > 0$)
Both roots in the interval **(p, q)**	$D \ge 0$ · $a \cdot f(p) > 0$ · $a \cdot f(q) > 0$ · $p < -b/2a < q$
Exactly one root in (p, q)	$f(p) \cdot f(q) < 0$



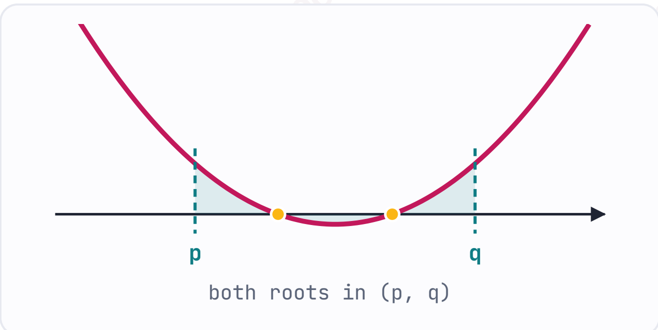
both roots < k



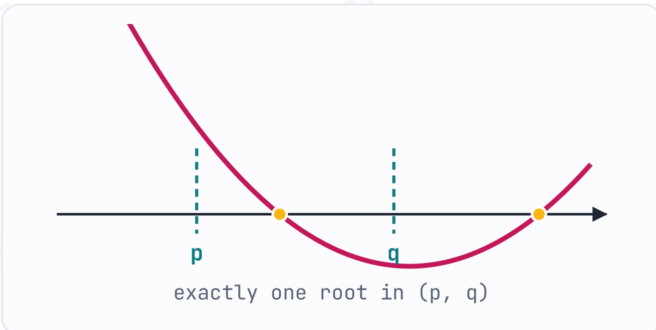
both roots > k



k lies between the roots



both roots in (p, q)



exactly one root in (p, q)

Each picture is the proof: the parabola ($a > 0$) sits so that the dashed reference values land where the conditions demand.

⚠ EDGE CASES THAT EXAMINERS LOVE

For **exactly one root in (p, q)** the clean condition $f(p)\cdot f(q) < 0$ assumes the endpoints are not themselves roots. If $f(p) = 0$ or $f(q) = 0$, handle that root separately and check the other root's location directly — do not just trust the product. Also, "between p and q" on JEE usually means the open interval; read whether endpoints are included.

WORKED · K BETWEEN THE ROOTS

Find all values of a for which 6 lies between the roots of $x^2 + 2(a-3)x + 9 = 0$.

Here the leading coefficient is $1 > 0$, so "6 between the roots" $\Leftrightarrow f(6) < 0$.

```
f(6) = 36 + 12(a-3) + 9 = 36 + 12a - 36 + 9 = 12a + 9
```

$f(6) < 0 \Rightarrow 12a + 9 < 0 \Rightarrow$ **$a < -3/4$**. (Because $a\cdot f(6) < 0$ with $a = 1$ already forces $D > 0$, no separate discriminant check is needed.)

WORKED · BOTH ROOTS IN AN INTERVAL

Find all m so that both roots of $x^2 - mx + (m+2) = 0$ lie in $(0, 3)$.

With $a = 1 > 0$, impose all four conditions:

```
D ≥ 0: m² - 4(m+2) ≥ 0 ⇒ m² - 4m - 8 ≥ 0 ⇒ m ≤ 2-2√3 or m ≥ 2+2√3
f(0) > 0: m + 2 > 0 ⇒ m > -2
f(3) > 0: 9 - 3m + m + 2 > 0 ⇒ 11 - 2m > 0 ⇒ m < 11/2
vertex in (0,3): 0 < m/2 < 3 ⇒ 0 < m < 6
```

Intersect everything. The vertex and $f(0)$ conditions give $0 < m < 11/2$; combined with $D \geq 0$ the only surviving piece is **$2+2\sqrt{3} \leq m < 11/2$** (since $2+2\sqrt{3} \approx 5.46 < 5.5$). Always intersect; never union.

3 Range of a Quadratic Expression

The range answers: as x runs over its allowed set, what values can $y = f(x)$ take? Two settings dominate JEE — the quadratic over all of R, and rational/constrained expressions handled by the discriminant method.

★ **RANGE OVER ALL REAL X — USE THE VERTEX**

The extreme value of $f(x) = ax^2+bx+c$ occurs at the vertex $x = -b/2a$, and equals $f(-b/2a) = $ **$-D/4a$** ($= c - b^2/4a$). Then:

- $a > 0$ (opens up): minimum $-D/4a$, range = **$[-D/4a, \infty)$**.
- $a < 0$ (opens down): maximum $-D/4a$, range = **$(-\infty, -D/4a]$**.

The vertex value −D/4a is the single boundary of the range; the parabola sweeps everything on one side of it.

▲ **THE DISCRIMINANT METHOD — FOR RATIONAL EXPRESSIONS**

To find the range of $y = (a_1x^2+b_1x+c_1)/(a_2x^2+b_2x+c_2)$, set y equal to the expression and *cross-multiply* to get a quadratic in x with y as a parameter. Since x must be real, impose $D_x \geq 0$; solving this inequality in y gives the range. Always check the value of y that makes the x^2 coefficient vanish separately, since at that y the equation is linear, not quadratic.

WORKED · RANGE BY DISCRIMINANT METHOD

Find the range of $y = (x^2 - x + 1)/(x^2 + x + 1)$, $x \in R$.

The denominator x^2+x+1 has $D = 1-4 < 0$ and positive leading coefficient, so it is always positive — no value of x is excluded. Cross-multiply:

```
y(x²+x+1) = x²-x+1 ⇒ (y-1)x² + (y+1)x + (y-1) = 0
```

For real x we need $D_x \geq 0$:

```
(y+1)² - 4(y-1)² ≥ 0 ⇒ [(y+1) - 2(y-1)][(y+1) + 2(y-1)] ≥ 0
⇒ (3 - y)(3y - 1) ≥ 0 ⇒ 1/3 ≤ y ≤ 3
```

Check the special value $y = 1$ (makes the x^2 term vanish): the equation becomes $2x = 0 \Rightarrow x = 0$, which is valid, and $y = 1$ already lies in $[1/3, 3]$. Hence the range is **[1/3, 3]**.

WORKED · RANGE ON A RESTRICTED DOMAIN

Find the range of $f(x) = x^2 - 4x + 3$ on the closed interval $[0, 3]$.

Vertex at $x = -b/2a = 2$, which lies inside $[0, 3]$, so the minimum is $f(2) = 4 - 8 + 3 = -1$. The maximum is at an endpoint: $f(0) = 3$, $f(3) = 0$, so the maximum is 3. Range = **[−1, 3]**.

Lesson: on a restricted domain always compare the vertex value (only if the vertex is inside the interval) with both endpoint values. Never assume the vertex is attainable.

4 Applications & JEE Patterns

These mixed patterns recombine the three tools above. Recognising which tool a question wants is the skill being tested.

PATTERN MAP — WHAT THE QUESTION IS REALLY ASKING

- "for all x" / "always positive" → sign of a with $D < 0$.
- "k between the roots" / "one root in (p,q)" → sign of $a\cdot f(k)$ or $f(p)f(q)$.
- "both roots in (p,q)" / "positive roots" → D, endpoint signs, vertex position together.
- "greatest / least value" / "range" → vertex $-D/4a$ (and endpoints if restricted).
- "common root" → eliminate to get the condition, then back-substitute.

WORKED · BOTH ROOTS POSITIVE

For what values of a does $x^2 - (a-3)x + a = 0$ have both roots positive?

Three conditions for both roots > 0 (with $a_{lead} = 1$): $D \geq 0$, sum > 0, product > 0.

```
product αβ = a > 0 ⇒ a > 0
sum α+β = a-3 > 0 ⇒ a > 3
D ≥ 0: (a-3)² - 4a ≥ 0 ⇒ a² - 10a + 9 ≥ 0 ⇒ a ≤ 1 or a ≥ 9
```

Intersection: $a > 3$ and ($a \leq 1$ or $a \geq 9$) gives $\mathbf{a \geq 9}$.

WORKED · MAXIMISE A CONSTRAINED EXPRESSION

If x is real, find the maximum value of $y = -2x^2 + 8x + 3$.

Here $a = -2 < 0$, so there is a maximum equal to $-D/4a$.

```
D = 8² - 4(-2)(3) = 64 + 24 = 88
max = -D/4a = -88 / (4·-2) = -88/-8 = 11
```

So the maximum value is **11**, attained at $x = -b/2a = -8/(2\cdot -2) = 2$.

WORKED · COMMON ROOT

If the equations $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ ($b \neq c$) have a common root, find a relation between b and c.

Subtract the two equations to eliminate x^2:

$$(b - c)x + (c - b) = 0 \Rightarrow (b - c)(x - 1) = 0$$

Since $b \neq c$, the common root is $x = 1$. Substituting into the first equation: $1 + b + c = 0$, so $\mathbf{b + c = -1}$. (For a common root, eliminating the highest power is almost always the fastest route.)

▲ SPEED HABITS FOR THE EXAM

Sketch the parabola first — even a rough up/down with root marks — before writing inequalities; the picture tells you which conditions you need and stops sign errors. For "location" problems, write the three checks (D, a·f at the reference points, vertex side) as a checklist and intersect. For "always positive/negative", the answer is two short conditions, never more. For range, ask first whether the domain is all of R (use the vertex) or restricted (compare vertex with endpoints).

★ Quadratics — One-Page Formula Sheet

Every condition on one page. Print this alone.

BASICS

$f(x)=ax^2+bx+c,\ a\neq 0$
$D = b^2-4ac$
roots $= (-b\pm\sqrt{D})/2a$
vertex $= (-b/2a,\ -D/4a)$

ROOT NATURE

$D>0$: two distinct real
$D=0$: one repeated real
$D<0$: complex conjugate
D = perfect square & $a,b,c \in \mathbb{Q}$ → rational roots

SUM & PRODUCT

$\alpha+\beta = -b/a$
$\alpha\beta = c/a$
$\alpha-\beta = \sqrt{D}/|a|$
form eqn: $x^2-(\text{sum})x+(\text{prod})=0$

SIGN OF F(X)

$D<0$ → sign of f = sign of a (all x)
$D>0$ → opposite to a between roots,
same as a outside roots
$>0\ \forall x$: $a>0$ & $D<0$

BOTH ROOTS < K

$D \geq 0$
$a\cdot f(k) > 0$
$-b/2a < k$

BOTH ROOTS > K

$D \geq 0$
$a\cdot f(k) > 0$
$-b/2a > k$

K BETWEEN ROOTS

$a\cdot f(k) < 0$
(forces $D > 0$ automatically)

BOTH ROOTS IN (P,Q)

$D \geq 0$
$a\cdot f(p) > 0,\ a\cdot f(q) > 0$
$p < -b/2a < q$

EXACTLY ONE IN (P,Q)

$f(p)\cdot f(q) < 0$
(check endpoints aren't roots)

BOTH ROOTS POSITIVE

$D \geq 0$
sum $= -b/a > 0$
product $= c/a > 0$

RANGE OVER R

extreme $= -D/4a$ at $x=-b/2a$
$a>0$: $[-D/4a,\ \infty)$
$a<0$: $(-\infty,\ -D/4a]$

RANGE (RATIONAL)

set y = expr, cross-multiply
quadratic in x → $D_x \geq 0$
solve inequality in y
check y that kills x^2 term

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